

# On the “Equivalence” of the Maxwell and Dirac Equations

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It is shown that Maxwell's equation cannot be put into a spinor form that is equivalent to Dirac's equation. First of all, the spinor  $\psi$  in the representation  $\vec{F} = \psi\bar{u}\psi$  of the electromagnetic field bivector depends on only three independent complex components whereas the Dirac spinor depends on four. Second, Dirac's equation implies a complex structure specific to spin 1/2 particles that has no counterpart in Maxwell's equation. This complex structure makes fermions essentially different from bosons and therefore insures that there is no physically meaningful way to transform Maxwell's and Dirac's equations into each other.

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**KEY WORDS:** Maxwell equation; Dirac equation; Lanczos equation; fermion; boson.

## 1. INTRODUCTION

The conventional view is that spin 1 and spin 1/2 particles belong to distinct irreducible representations of the Poincaré group, so that there should be no connection between the Maxwell and Dirac equations describing the dynamics of these particles.

However, it is well known that Maxwell's and Dirac's equations can be written in a number of different forms, and that in some of them these equations look very similar (e.g., Fushchich and Nikitin, 1987; Good, 1957; Kobe, 1999; Moses, 1959; Rodrigues and Capelas de Oliveira, 1990; Sachs and Schwebel, 1962). This has led to speculations on the possibility that these similarities could stem from a relationship that would be not merely formal but more profound (Campolattaro, 1990, 1997), or that in some sense Maxwell's and Dirac's equations could even be “equivalent” (Rodrigues and Vaz, 1998; Vaz and Rodrigues, 1993, 1997).

The purpose of this paper is to investigate these possibilities and to give some arguments confirming that these formal similarities cannot lead to a physically meaningful identification of Maxwell's and Dirac's equations. To facilitate this

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investigation, and to make all calculations explicit, Lanczos's biquaternionic formulation of Maxwell's and Dirac's equations will be used (Gspöner and Hurni, 1998, 2001; Lanczos, 1929). The mathematical advantages of this formalism (which uses only complex numbers and the quaternion algebra) is that it is irreducible in the sense that, compared to formulations using larger Clifford algebras, the number of explicit components, symbols, and operations is minimal. Moreover, like all formulations based on Clifford algebras, most calculations are in general simpler than with the standard formulations based on tensors, spinors, and matrices.

## 2. THE ELECTROMAGNETIC FIELD AND MAXWELL'S EQUATION IN SPINOR FORM

The starting point of Campolattaro's and Rodrigues's formulations of Maxwell's equation is to write the electromagnetic field in spinor form using the standard  $\gamma$ -matrices formalism (Campolattaro, 1990)

$$F^{\mu\nu} \equiv \bar{\Psi} S^{\mu\nu} \Psi, \quad (1)$$

or the Clifford bundle formalism of Vaz and Rodrigues (1993, 1997)

$$\mathcal{F} \equiv \psi \gamma_{21} \psi^{\sim} \quad (2)$$

where the involution  $(\ )^{\sim}$  is the reversion operation in the Clifford algebra.

Maxwell's first and second equations are then written

$$\partial_{\mu} \bar{\Psi} \gamma^{\mu} S^{\mu\nu} \Psi = 0, \quad \partial_{\mu} \bar{\Psi} S^{\mu\nu} \Psi = j^{\mu}, \quad (3)$$

or, respectively,

$$\partial \psi \gamma_{21} \psi^{\sim} - (\partial \psi \gamma_{21} \psi^{\sim})^{\sim} = 0, \quad \partial \psi \gamma_{21} \psi^{\sim} + (\partial \psi \gamma_{21} \psi^{\sim})^{\sim} = 2\mathcal{J}. \quad (4)$$

Equations (3) and (4) are strictly equivalent to Maxwell's equations. However, the spinors  $\Psi$  or  $\psi$  are *not* equivalent to a Dirac spinor because they have only six independent real components while the Dirac spinor has eight. To see this explicitly, we rewrite (2) in the biquaternion formalism

$$\vec{F} = \vec{E} + i\vec{B} \equiv \psi \vec{u} \bar{\psi} = \rho e^{i\beta} \vec{\mathcal{L}} \vec{u} \vec{\mathcal{L}} \quad (5)$$

where  $\vec{u}$  is a constant unit vector,  $\vec{\mathcal{L}}$  a unit biquaternion, and the complex factor  $\rho e^{i\beta}$  corresponds to a duality transformation (Rainich, 1925). This representation is general because any non-null electromagnetic field can always be obtained by means of a duality transformation and of a Lorentz transformation  $\mathcal{L}(\ )\vec{\mathcal{L}}$  from a reference frame in which the electric and magnetic field vectors are parallel (Landau and Lifshitz, 1975, see Sections 24 and 25; Misner *et al.*, 1970, see Exercise 20.7).

However, expressions (1), (2), and (5) are invariant under any gauge transformation of  $\Psi$ ,  $\psi$ , or  $\psi$  which commutes with  $S^{\mu\nu}$ ,  $\gamma_{21}$ , or  $\vec{u}$ . For instance, the

substitution  $\mathcal{L} \rightarrow \mathcal{L} \exp(c\vec{u})$  with  $c \in \mathbb{C}$  leaves (5) invariant. Therefore, the effective Lorentz transformation in (5) depends not on six but just on four parameters. This can be seen explicitly by solving (5) for  $\mathcal{L}$ . It comes

$$\mathcal{L} = \frac{\vec{f} + \vec{u}}{\sqrt{2(1 + \vec{f} \cdot \vec{u})}} \tag{6}$$

where  $\vec{f}$  is the unit vector such that  $\vec{F} = \rho e^{i\beta} \vec{f}$ . This expression, which was first derived by Gürsey (1956, p. 167), confirms that the spinor  $\psi = \sqrt{\rho} e^{i\beta/2} \mathcal{L}$  associated with an electromagnetic field has three complex components, and not four like a Dirac spinor.

### 3. MAXWELL’S EQUATION IN DIRAC-LIKE FORM

By a number of lengthy tensor manipulations Campolattaro (1990) succeeded in reducing equations (2) to a single nonlinear equation in which the four-gradient of the spinor  $\Psi$  appears on one side, and in which the nonlinearity appears on the other side as a complicated variable factor that would be the mass if  $\Psi$  was a Dirac field. The same calculation was repeated by Vaz and Rodrigues (1993) who confirmed the power of the Clifford number formalism by deriving an equivalent equation in a very straightforward manner. In the formalism of Rodrigues (1997, 1998), Maxwell’s equations (4) are equivalent to the nonlinear equation

$$\partial\psi\gamma_{21} = \frac{\exp(\gamma_5\beta)}{\rho} \left[ \frac{1}{2}\mathcal{J} + (j + \gamma_5g) \right] \psi \tag{7}$$

where  $j = \gamma^\mu \langle (\partial_\mu \psi) \gamma_{21} \psi \tilde{\ } \rangle$  and  $g = \gamma^\mu \langle (\partial_\mu \psi) \gamma_5 \gamma_{21} \psi \tilde{\ } \rangle$ . In the case where  $\mathcal{J} = 0$ , and provided that  $\rho$ ,  $\beta$ ,  $j$ , and  $g$  are constants, this expression would be similar to Dirac’s equation, which in Rodrigues’s formalism is

$$\partial\psi_D\gamma_{21} = m\psi_D. \tag{8}$$

However, since  $\psi$  in (7) has only six real functions in its components, it cannot be made equivalent to (8) in the general case where the Dirac spinor  $\psi_D$  has eight independent components. For the same reason, contrary to what Campolattaro and Rodrigues tried to do, it is not possible to find nontrivial cases in which the mass term in Maxwell’s equation in spinor form becomes a constant. In particular, it is not possible to use constraints such as  $\partial \cdot j = 0$  and  $\partial \cdot g = 0$ , because they reduce the number of independent real components from six to four.

#### 4. INTRINSIC DIFFERENCE BETWEEN MAXWELL'S AND DIRAC'S EQUATIONS

Let us translate Eqs. (4) and (8) into the biquaternion formalism (Gsponer and Hurni, 1998, 2001). Maxwell's equations are then

$$\nabla \vec{F} - \vec{F} \sim \nabla = 0, \quad \nabla \vec{F} + \vec{F} \sim \nabla = 2J, \tag{9}$$

and Dirac's equation becomes the *Dirac-Lanczos equation* (Lanczos, 1929)

$$\bar{\nabla} \psi_D = im \psi_D^* \vec{u}. \tag{10}$$

Both equations are of first order, which is why the question of their possible "equivalence" can arise. However, there is one essential difference: Dirac's equation relates the field  $\psi_D$  to its complex conjugate  $\psi_D^*$ , while Maxwell's equation (as well as Proca's equation for a massive spin 1 particle) do not. This complex conjugation operation arises naturally when the spin 1/2 and spin 1 field equations are consistently derived from Lanczos's fundamental equation (Gsponer and Hurni, 1998, 2001), and it is intimately connected to the fact that fermions are essentially different from bosons (Gsponer and Hurni, 1994). Indeed, by studying the time-reversal transformation of (10), one finds  $T^2 = -1$ , which implies the Pauli exclusion principle and Fermi statistics (Feynman, 1987; Weinberg, 1998, p. 80).

Therefore, since the spinor representation of the electromagnetic field (5) involves  $\psi$  and  $\bar{\psi}$ , not  $\psi$  and  $\psi^*$ , it is impossible to transform Maxwell's equations (9) by any algebraic manipulation to get a Dirac equation for the spinor  $\psi$ , even if we accept a nonlinear variable mass term.

The complex conjugation operation that is explicit in (10) has a counterpart in any formulation of Dirac's equation. Its absence in the standard  $\gamma$ -matrices formulation or in other Clifford formulations of Dirac's equation, e.g. (8), needs therefore to be explained. The reason is that these formulations either use a larger algebra, so that the complex conjugation in (10) can be avoided, or a "complex structure" (or "complex geometry") by which operations on complex numbers are replaced by algebraic operations on real numbers (De Leo, 2001; This is a recent example in which ordinary complex numbers are replaced by linear functions of real quaternions).

For example, in the standard  $\gamma$ -matrices formulation the Dirac algebra  $\mathbb{D} \sim \mathcal{M}_4(\mathbb{C})$  is used instead of the biquaternion algebra  $\mathbb{B} \sim \mathcal{M}_2(\mathbb{C})$ . Then, the complex conjugate of the Dirac matrices is given by  $\gamma_\mu^* = \gamma_0 \gamma_\mu^T \gamma_0$  where  $( )^T$  denotes transposition, and the "Dirac adjoint"  $\bar{\Psi} \equiv \Psi^\dagger \gamma_0$  is required instead of  $\Psi^\dagger$  in scalar products (see, e.g., Weinberg (1998)). Similarly, Lanczos's "complex" formulation of Dirac's equation (10) can be transformed into the "real" formulation of Hestenes (1966, see Sections 7 and 13) by using the identification  $D^* = \gamma_0 D \gamma_0$  to replace the symbolic complex conjugation operator  $( )^*$  by the linear function  $\gamma_0 ( ) \gamma_0$  (see, in particular, Fauser, 2001, where it is shown that this function provides a link between seemingly inequivalent formulations of Dirac's equation).

## 5. CONCLUSION

By various algebraic transformations it is possible to put Maxwell’s equation into a form that is very similar to Dirac’s equation. Similarly, it is also possible to replace the complex conjugation operator that is explicit in the Dirac–Lanczos equation (10) by an algebraic operation that effectively implements this complex conjugation. However, the transformed Maxwell equation will still depend on only six real functions while Dirac’s equation requires eight, and the complex structure which makes a Dirac particle a fermion rather than a boson will remain an essential feature of Dirac’s equation.

Indeed, the complex structure inherent to Dirac’s equation corresponds to the fact that the Dirac field is only meaningful in the context of quantum mechanics where complex numbers are essential. On the other hand, such a complex structure is absent in Maxwell’s equation so that Maxwell’s field has a consistent interpretation in the context of classical electrodynamics where complex numbers are not essential.

There are of course other fundamental differences between the Maxwell and Dirac equations and their interpretations that are not discussed in this paper (see, e.g., Good (1957)).

Finally, while practical calculations are made most easily using a formulation such as the standard  $\gamma$ -matrices, it is important to stress that the Dirac–Lanczos equation (10) has the considerable didactical advantage to make manifest the fermionic complex structure that is hidden in Dirac’s original formulation, as well as in most other ones, e.g., Rodrigues’s (8), or Hestenes’s (Hestenes, 1966).

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